The Twentieth Annual CNU Regional High School Mathematics Contest

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1. How many distinct prime factors does 2020 have?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

2. How many unordered pairs $\{a, b\}$ can one select from $\{1, 2, \ldots, 20\}$ such that $|a - b| = 4$?

   (A) 16 (B) 18 (C) 20 (D) 22 (E) 24

3. Take a rectangle and decrease its length by 2 and increase its width by 6 to form a square of area 289. Find the perimeter of the original rectangle.

   (A) 60 (B) 64 (C) 68 (D) 72 (E) 76

4. Two six-sided dice each have two red, two yellow, and two blue faces. If we roll the dice, what is the probability that both dice show matching colors?

   (A) $\frac{1}{36}$ (B) $\frac{1}{12}$ (C) $\frac{1}{9}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

5. Using pennies and/or dimes, how many ways are there to make \$20.20?

   (A) 200 (B) 201 (C) 202 (D) 203 (E) 204

6. It takes three lumberjacks three minutes to saw three logs into three pieces each. How many minutes does it take six lumberjacks to saw six logs into six pieces each? Assume each cut takes the same amount of time, with one lumberjack assigned to each log.

   (A) 3 (B) 6 (C) $7\frac{1}{2}$ (D) 12 (E) 15

7. The base seven number 4231 is equivalent to what base ten number?

   (A) 1215 (B) 1492 (C) 1666 (D) 1865 (E) 2020

8. If you answer the first three questions of this contest by picking a choice at random, then you have a better than 50% chance of getting exactly how many of those three questions correct?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

9. For $x > 0$, how many solutions does the equation $\log_{10}(x + \pi) = \log_{10} x + \log_{10} \pi$ have?

   (A) 0 (B) 1 (C) 2 (D) more than 2 but finitely many (E) infinitely many

10. A 6-sided die is rolled; then a 10-sided die is rolled. What is the probability that the two numbers appear in strictly increasing order? (Note: A roll of 3-9 is strictly increasing; a roll of 3-3 is not strictly increasing.)

(A) $\frac{11}{30}$ (B) $\frac{5}{12}$ (C) $\frac{3}{5}$ (D) $\frac{13}{20}$ (E) $\frac{7}{10}$

11. A game is played where teams gain either 4 points or 7 points when they score. No other point values are possible. What is the highest total score that cannot be obtained?

(A) 12 (B) 17 (C) 69 (D) 151 (E) No highest value exists

12. If $\sin\alpha = -\frac{\sqrt{2}}{2}$ and $\cos(\alpha-\beta) = \frac{1}{2}$ with $\beta > 0$, what is the minimum value of β?

(A) $\frac{\pi}{24}$ (B) $\frac{\pi}{18}$ (C) $\frac{\pi}{12}$ (D) $\frac{\pi}{6}$ (E) $\frac{\pi}{4}$

13. You have two pouches. The first pouch contains three blue and two red balls; the second pouch contains two blue and three red balls. You shuffle the pouches and select one. From the selected pouch, your friend draws a red ball. What is the probability that the selected pouch is the one with three blue balls?

(A) $\frac{1}{5}$ (B) $\frac{6}{25}$ (C) $\frac{3}{10}$ (D) $\frac{2}{5}$ (E) $\frac{1}{2}$

14. A cubic polynomial $p(x)$ with leading coefficient 1 has three real roots. The median of these roots is 0, as is the average. If the range of the set of roots is 10, what is the coefficient of x in $p(x)$?

(A) -25 (B) -10 (C) -5 (D) 0 (E) Cannot be determined

15. When written out, how many digits does 20^{20} have? (22, **27**, 32, 37, 42)

(A) 22 (B) 27 (C) 32 (D) 37 (E) 42

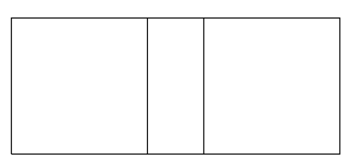
16. How many ordered pairs of positive integers (a, b) satisfy $a^2 - b^2 = 20$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

17. The large rectangle below has been divided into two squares and a smaller rectangle. The large and small rectangles are similar. If the shorter side of the large rectangle has length 1 unit, then how many units long is the large rectangle's longer side?

(A) $1+\sqrt{2}$ (B) $2\sqrt{2}$ (C) $\sqrt{5}$

(D) $\sqrt{10}-1$ (E) $\sqrt{12}-1$



18. Let $|x|$ represent the absolute value of x. What may be said concerning the solution(s) of

$$|x|^2 + |x| - 11 = 0?$$

(A) There are no solutions.
(B) The sum of the solutions is 1.
(C) The sum of the solutions is 0.
(D) The product of the solutions is -11.
(E) There is only one solution.

19. Compute $\sqrt{20 + \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}}$.

(A) 5 (B) 6 (C) $3\sqrt{5}$ (D) 7 (E) $4\sqrt{5}$

20. A cylindrical can just holds three tennis balls stacked on top of each other (so they touch the sides, top and bottom of the can). The height of the stacked balls is 9 inches. What percent of the can is empty space outside of the balls?

(A) $4\frac{1}{2}$ % (B) 15% (C) 27% (D) 31% (E) $33\frac{1}{3}$ %

21. What is the value of the following product?

$$2^{2020} \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{16}\right) \cos\left(\frac{\pi}{32}\right) \cdots \cos\left(\frac{\pi}{2^{2020}}\right)$$

(A) $2^{-2020} \sin\left(\frac{\pi}{2^{2019}}\right)$
(B) $\tan\left(\frac{\pi}{2^{2019}}\right)$
(C) $2 \csc\left(\frac{\pi}{2^{2020}}\right)$
(D) $4 \sec\left(\frac{\pi}{2^{2019}}\right)$
(E) $2^{2019} \cot\left(\frac{\pi}{2^{2020}}\right)$

22. Let $f(x) = 1/(1-x)$. Define $f_1(x) = f(x)$ and $f_{n+1}(x) = f(f_n(x))$ for $n \geq 1$. What is $f_{2020}(x)$?

(A) x (B) $\dfrac{1}{1-x}$ (C) $\dfrac{x-1}{x}$ (D) $\dfrac{x}{x-1}$ (E) $\dfrac{x}{1-x}$

23. Let a be a positive real number such that

$$\log_a(2020) + \log_a(2020^2) + \log_a(2020^3) + \cdots + \log_a(2020^{20}) = 630.$$

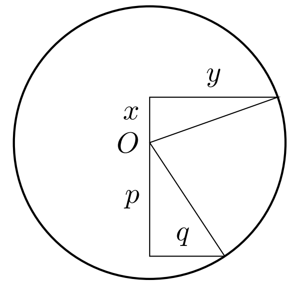
What is a?

(A) $\sqrt[3]{2020}$ (B) e^3 (C) 2020 (D) 6060 (E) $2020^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{20}}$

24. In the figure, the two triangles are right triangles with sides of length x, y, p, and q as shown. The shared point O is the center of the circle and the indicated vertices are on the circle. Given that $x^2 + y^2 + p^2 + q^2 = 72$, find the circumference of the circle.

(A) 8π (B) 9π (C) 12π

(D) 24π (E) 36π



25. For each real number x, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x. For how many positive integers n is it true that $n < 2020$ and $\lfloor \log_2 n \rfloor$ is a positive even integer?

(A) 1,334 (B) 1,335 (C) 1,336 (D) 1,337 (E) 1,338

26. How many five-digit numbers greater than 10,000 begin with an odd digit and have no digit equal to 0, end with an even digit and have no digit equal to 0, or have all digits identical?

(A) 40,824 (B) 44,469 (C) 47,385 (D) 59,049 (E) 65,610

27. A right circular cone has base radius r and height h. The cone lies on its side on a flat table. As the cone rolls on the surface of the table without slipping, the point where the cone's base meets the table traces a circular arc centered at the point where the vertex touches the table. The cone first returns to its original position on the table after making 17 complete rotations. The value of h/r can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

28. What is the maximum value of the following function?

$$f(x) = \frac{\sin^3 x \cos x}{\tan^2 x + 1}$$

(A) 1/8 (B) 1/4 (C) 1/3 (D) 1/2 (E) 1

29. The digits of 2020 sum to 4. How many positive integers less than 2020 also have digits that sum to 4?

(A) 25 (B) 27 (C) 29 (D) 31 (E) 33

30. If $x - \frac{1}{x} = 5$, what is $x^4 + \frac{1}{x^4}$?

(A) 531 (B) 577 (C) 625 (D) 675 (E) 727

31. Consider a right triangle with acute angles A and B. Find the maximum value of $(\sin A)(\sin B)$.

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$ (E) 1

32. Two perpendicular chords of a circle intersect at point P. One chord is 7 units long, divided by P into segments of length 3 and 4, while the other chord is divided into segments of length 2 and 6. What is the diameter of the circle?

(A) $\sqrt{56}$ (B) $\sqrt{65}$ (C) $\sqrt{75}$ (D) $\sqrt{79}$ (E) $\sqrt{89}$

33. There are 480 flies sitting on the squares of a 16×30 board, with one fly in each square. We say that two flies are neighbors if their squares share a side. They take off, and then land on a 15×32 board, again with one fly in each square. What is the probability that each fly has every neighbor the same on both boards?

(A) 0 (B) $\frac{1}{230,400}$ (C) $\frac{1}{120}$ (D) $\frac{1}{20}$ (E) $\frac{31}{256}$

34. For what values of k does the equation $\log_{10}(kx) = 2\log_{10}(x+1)$ have exactly one real solution?

(A) $k < -1$ or $k = 2$ (B) $k < 0$ or $k = 2$ (C) $k < 1$ or $k = 2$
(D) $k < 0$ or $k = 4$ (E) $k < 0$ or $k = 8$

35. First x is chosen at random from the set $\{1, 2, 3, \ldots, 99, 100\}$, and then y is chosen at random from the same full set. The probability that the integer $3x + 7y$ has units digit 8 is:

(A) $\frac{1}{8}$ (B) $\frac{1}{10}$ (C) $\frac{3}{16}$ (D) $\frac{1}{5}$ (E) $\frac{1}{4}$

36. Given that $f(x) = x^4 - ax^2 - bx + 2$ is divisible by $x^2 + 3x + 2$, find $a + b$.

(A) 0 (B) 3 (C) 5 (D) 7 (E) 9

37. Simplify $\sqrt[3]{\sqrt{50}+7} - \sqrt[3]{\sqrt{50}-7}$.

(A) 1 (B) $\sqrt[3]{2}$ (C) $\sqrt{2}$ (D) 2 (E) $2-\sqrt{2}$

38. Points A and B lie on the edge of a circular lake. How long to the nearest minute does it take Rachel to oar from A to B if the diameter of the lake is $2\sqrt{3}$ and she rows 3 mph? (In the picture, $\overline{AC}$ is a diameter.)

(A) 17 min. (B) 35 min. (C) 40 min. (D) 60 min. (E) 80 min.

39. A circle of radius 1 in rolls around *inside* a circle of radius 12 in. How many full revolutions does the inside circle make before returning to its original position?

(A) 5 (B) 6 (C) 11 (D) 12 (E) 24

40. Let x_1, x_2, x_3 be nonnegative real numbers such that $x_1+x_2+x_3=1$. Calculate the maximum value of

$$(x_1+3x_2+5x_3)(x_1+\frac{x_2}{3}+\frac{x_3}{5})$$

(A) $\frac{16}{15}$ (B) $\frac{5}{4}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$ (E) $\frac{9}{5}$

Answer Key

1. B
2. A
3. A
4. D
5. D
6. C
7. B
8. A
9. B
10. D
11. B
12. C
13. D
14. A
15. B
16. A
17. A
18. E
19. A
20. E
21. C
22. B
23. A
24. C
25. C
26. B
27. C
28. A
29. B
30. E
31. C
32. B
33. A
34. D
35. B
36. E
37. D
38. D
39. C
40. E